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Thermoelectric magnetohydrodynamics

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Thermoelectric currents in the presence of magnetic fields can cause pumping or stirring of liquid-metal coolants in nuclear reactors or stirring of molten metal in industrial metallurgy. The interaction between the thermal and magnetohydrodynamic fields is a mutual one owing to alterations in the thermal convection and to the Peltier and Thomson effects (although these are usually small). This paper sets up the equations of magnetohydrodynamics and thermal convection when coupled by thermoelectricity and solves some illustrative problems in which the thermal field is known *ab initio*. Examples where the effects are due to either continuous or discontinuous variation of material composition are included. Practical magnitudes are discussed for the case of a fusion-reactor blanket, where the effects are potentially important owing to the unusual thermoelectric power of lithium.

1. Introduction

First-generation fusion reactors will probably react deuterium (D) with tritium (T), which must be bred in a blanket from the irradiation of lithium by the neutrons excaping from the reactors. As most of the energy released in the DT fusion reaction is borne by those neutrons, the blanket is the main source of the heat to be used in the associated heat engine. It is therefore natural to consider using the lithium (which melts at 180 °C) as the coolant which conveys this heat to the heat engine.

The main difficulty with this proposal as applied to magnetic confinement systems is that pumping the lithium across the high magnetic fields which are envisaged, especially in Tokamak reactors, requires pressures which could take the materials used for the pipework beyond their creep limits, not to mention the power degraded in pumping. (See, for instance, Hunt & Hancox 1971; Hancox & Booth 1971, Stanbridge *et al.* 1974.) The blanket has to be immersed in the magnetic field because it protects the windings from neutron bombardment.

It so happens that lithium is unusual in having a high and positive absolute thermoelectric power. This raises the possibility that, wherever sufficiently high temperature gradients occur (or are deliberately promoted) and the lithium is in contact with a solid conductor of markedly different thermoelectric power, preferably negative, electric currents will circulate. In conjunction with the prevailing magnetic field, these could strongly affect the motions in the lithium. Vigorous stirring might result, rendering wholly invalid any estimates of temperature distributions arrived at by assuming static lithium (the natural assumption in the presence of a strong magnetic field). This is a matter of some concern because the problems of thermal stress, expansion and creep will have to be fully mastered for successful design of fusion-reactor blankets. More interesting, it is conceivable that the fluid could be encouraged to pump *itself* across the magnetic field without excessive pressure changes, thereby overcoming the main objection to the use of the liquid lithium as a coolant.

This paper is a preliminary survey of the theory of thermoelectric magnetohydrodynamics (TEMHD) and of some of the simpler problems that arise. It appears to be a largely unexplored field of fluid mechanics, having potential applications in industrial metallurgy as well as fusion technology.

There have been several papers on TEMHD pumping, but none of these has explored the detailed fluid mechanics. Most of the work was done in the context of fast fission reactors, cooled by liquid metals (Murgatroyd 1951; Luebke & Vandenberg 1954; Rex 1961; Osterle & Angrist 1964; Perlow & Davis 1965; de Cachard & Caunes 1969; Makarov & Cherkasskii 1969). In several of these papers the TEMHD effect was to be enhanced by the use of semi-conductors such as lead telluride, or well-matched pairs of solid metals such as chromel/constantan, with the liquid metal playing a passive (isothermal) role from the thermoelectric point of view. In this paper we concentrate attention mainly on cases where the thermocouple action is due purely to the liquid metal and a single solid metal, which forms the ducting. The design constraints on a fusion-reactor blanket would appear to discourage the use of more elaborate combinations.

Section 2 presents the basic theory of TEMHD and shows how the extra fluid-mechanical effect operates via the boundary conditions in most cases. Section 3 presents data on the magnitude of the effects in lithium and discusses in order-of-magnitude terms their practical implications. Section 4 outlines the ways in which some simple problems in MHD duct flow are modified by thermoelectric action at the wall/fluid interface while § 5 discusses the possibility of applications to industrial metallurgy and solves a typical problem, which also serves to exemplify those problems in which the effects are due to *continuous* variation of material composition, in contrast to sudden changes at an interface.

2. The theory of TEMHD

As formulations of thermoelectricity in field-theory terms are not widely available, a fairly full discussion is given.

To express the fact that conduction current, of intensity $\mathbf{j}$, can be caused by temperature gradient $\operatorname{grad} T$ as well as by an electric field $\mathbf{E}$ or an e.m.f. $\mathbf{v}\times\mathbf{B}$ due to motion at velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$, Ohm's law must be generalized to

$$\mathbf{j}/\sigma = \mathbf{E}+\mathbf{v}\times\mathbf{B}-S\operatorname{grad} T, \tag{1}$$

in which S is called the *absolute thermoelectric power* of the conducting medium in question, and σ is its electrical conductivity measured under isothermal conditions. A Hall term could be added, but this is not necessary in applications in liquid metals.

In a stationary medium of *uniform composition* under an irrotational electric field (magnetic induction due to $\partial\mathbf{B}/\partial t$ being absent), no current is observed to flow, whatever the distribution of T, for $S \operatorname{grad} T$ is also irrotational because S is here a function of T only and $\operatorname{grad} S\times\operatorname{grad} T$ vanishes. We may set

$$W(T) = \int S\,dT, \tag{2}$$

integrated from some datum temperature.

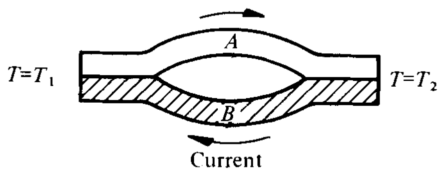


FIGURE 1. A simple thermocouple.

Then
$$\mathbf{j}/\sigma = \mathbf{E} - \operatorname{grad} W \tag{3}$$
$$= -\operatorname{grad}(V+W) = -\operatorname{grad} U, \quad \text{say}, \tag{4}$$

$\mathbf{E}$ being irrotational with potential V. In formulations of MHD and other problems where the temperature is non-uniform it is common practice to write only V in (4), but it should be appreciated that strictly V should be replaced by U, which includes the thermoelectric term W. Although confusion over the energetics can arise if V or $\mathbf{E}$ is used wrongly, ignoring thermoelectricity, this procedure usually causes no problems unless a medium of varying or different composition is introduced (e.g. by the insertion of potential probes or conducting walls or owing to concentration gradients in a diffusion boundary layer, etc.).

When the composition of the stationary medium or media varies from place to place, S grad T becomes rotational in general for grad S need no longer be parallel to grad T. Then currents must circulate even if the conductor is stationary and $\partial\mathbf{B}/\partial t$ is absent. The extreme case occurs when the composition and S vary discontinuously across an interface along which T varies. This is the key phenomenon in relation to TEMHD for homogeneous liquid metals flowing in conducting containers, for only at the interface are grad S and grad T not parallel. The thermoelectric e.m.f.'s are determined by the temperature distributions *along the interface*, irrespective of the temperatures elsewhere.

A familiar manifestation is the bimetallic thermocouple, shown in figure 1, with its two discrete points of contact on the interface between the two media A and B, the contacts being at different, specific temperatures, T_1 and T_2. A current flows around the circuit under the influence of the *Seebeck e.m.f.*, which equals

$$\mathscr{E}_{12} = \oint \frac{\mathbf{j}\,.\,d\mathbf{r}}{\sigma},$$

integrated round the circuit in the current-flow direction under conditions where $\partial\mathbf{B}/\partial t$ and $\mathbf{v}\times\mathbf{B}$ are absent. Then (1) indicates that

$$\mathscr{E}_{12} = -\int_{T_1}^{T_2} S_A \operatorname{grad} T\,.\,d\mathbf{r} + \int_{T_1}^{T_2} S_B \operatorname{grad} T\,.\,d\mathbf{r} = \int_{T_1}^{T_2} P\,dT, \tag{5}$$

where $P = S_B - S_A$, the thermoelectric power of the metal pair. Potentiometric measurements of the Seebeck e.m.f. cannot yield the values of the absolute powers S_B and S_A, only their difference. Note also that there is no discontinuity in the electric potential across each interface, if contact resistance and electrochemical effects are excluded.

Equation (5) is readily generalized to cases with more than two metals in the circuit, as commonly occurs with thermocouple instrumentation.

The other facet of thermoelectricity is the fact that electric current flow causes additional heat flow, where by 'heat' is meant all energy transport other than that described by the Poynting vector and the mechanical work transfer associated with the motion of stressed media. The heat flow intensity $\mathbf{Q}$ is given by a modified version of Fourier's law:

$$\mathbf{Q} = -K \operatorname{grad} T + ST\mathbf{j}, \tag{6}$$

in which K is the usual thermal conductivity, measured under conditions where $\mathbf{j}$ is absent. The formulation (6) is preferred to one involving $\mathbf{E}$ or grad V simply because of the way that K is defined. The coefficient S appears also in (6), making the current-driven part of the entropy-flow intensity $\mathbf{Q}/T$ equal to $S\mathbf{j}$, because of the Onsager reciprocal relations of irreversible thermodynamics. For a full discussion see Woods (1975), who relates it to the essential reversibility of the thermoelectric coupling, originally perceived by W. Thomson (Lord Kelvin) for an ideal thermocouple with $K = 0$ and $\sigma = \infty$.

S is also called the *entropy transport parameter*. Since $\mathbf{j}$ is in the opposite direction to the drift of conduction electrons, it might naïvely be expected that S would be negative for metals. This is true for most metals, but lithium, for which S is large and *positive*, is a conspicuous exception.

The *Thomson effect* follows from (6). Consider a current- and heat-conducting medium at rest under a given temperature distribution at a given instant. We exclude magnetization and ignore polarization energy. There is an electrical energy input to the medium at a rate $\mathbf{E}.\mathbf{j}$ per unit volume and time. Hence the rate at which energy is being stored locally in the medium is

$$\begin{aligned} \dot{W} &= -\operatorname{div} \mathbf{Q} + \mathbf{E}.\mathbf{j} \\ &= \{\operatorname{div}(K \operatorname{grad} T) - \operatorname{div}(ST\mathbf{j})\} + \mathbf{j}^2/\sigma + \mathbf{j}S.\operatorname{grad} T, \end{aligned} \tag{7}$$

if (1) and (6) are invoked. We take the normal, 'low-frequency' approximation that

$$\operatorname{div} \mathbf{j} = 0 \tag{8}$$

and then (7) leads to

$$\dot{W} = \mathbf{j}^2/\sigma + \operatorname{div}(K \operatorname{grad} T) - \mathbf{j}T.\operatorname{grad} S. \tag{9}$$

For a given local temperature distribution, the 'extra' local heating due to $\mathbf{j}$ exceeds the Joule heating $\mathbf{j}^2/\sigma$ by the final term $-\mathbf{j}T.\operatorname{grad} S$. Note that this effect still occurs even when $\mathbf{j}$ is due to e.m.f.'s other than thermoelectric ones or when S varies because of variable composition, provided $\mathbf{j}$ has a component parallel to grad S. The effect comes about because the variation of S in (6) means that a given current flux transports a varying entropy flux. The case where S varies discontinuously at an interface is discussed later.

In practice local temperatures do not stay the same when $\mathbf{j}$ is introduced so this effect shows itself instead as a change in grad T that enables the Joule heat to get away and allows for some of the heat being now transported by $\mathbf{j}$.

When the medium is of uniform composition and S is a function of T only, the extra final term in (9) can be written as

$$-T(dS/dT)\,\mathbf{j}.\operatorname{grad} T, \tag{10}$$

which describes the *Thomson* effect. The quantity $T(dS/dT)$ is called the Thomson coefficient.

Several other of the equations of MHD are unaffected by thermoelectric phenomena, namely

$$D\rho/Dt = -\rho \operatorname{div} \mathbf{v}, \tag{11}$$

$$\rho D\mathbf{v}/Dt + \operatorname{grad} p = \mathbf{j} \times \mathbf{B} + \text{visc.}, \tag{12}$$

$$-\partial \mathbf{B}/\partial t = \operatorname{curl} \mathbf{E}, \tag{13}$$

$$\mu \mathbf{j} = \operatorname{curl} \mathbf{B} \tag{14}$$

(which we shall ignore in this paper, always assuming that the magnetic Reynolds number is low and the field $\mathbf{B}$ known *ab initio*), and

$$\operatorname{div} \mathbf{B} = 0, \tag{15}$$

in which ρ = fluid density and p = fluid pressure. 'visc.' denotes standard viscous terms.

The energy and entropy equations *are* modified, however. The first law of thermodynamics for a travelling fluid element, expressed per unit volume, becomes

$$-\operatorname{div} \mathbf{Q} + \mathbf{E}.\mathbf{j} = (\operatorname{div} p\mathbf{v} + \text{visc.}) + \rho(D/Dt)(u + \tfrac{1}{2}\mathbf{v}^2), \tag{16}$$

in which 'visc.' denotes work done by viscous stresses and u is specific internal energy, related to the specific entropy s by the usual equation

$$T\,ds = du - (p/\rho^2)\,d\rho. \tag{17}$$

Inserting $\mathbf{E}$ from (1), $\operatorname{div} \mathbf{v}$ from (11), grad p from (12) and $\mathbf{Q}$ from (6) leads to the entropy equation

$$\rho T\, Ds/Dt = \operatorname{div}(K \operatorname{grad} T) + \mathbf{j}^2/\sigma - \mathbf{j}T.\operatorname{grad} S + \text{visc.}, \tag{18}$$

in which the term $-\mathbf{j}T.\operatorname{grad} S$ again appears. 'visc.' now denotes standard dissipative terms. If the effect of pressure changes on the entropy can be ignored, as is the case in most problems involving liquids, we may put $\rho T\,ds = C dT$, where C = volumetric heat capacity, and then

$$C\,DT/Dt = \operatorname{div}(K \operatorname{grad} T) + \mathbf{j}^2/\sigma - \mathbf{j}T.\operatorname{grad} S + \text{visc.}, \tag{19}$$

the heat convection equation. Note that the grad S term can in principle be active even when $\mathbf{j}$ is due to non-thermoelectric e.m.f.'s. Generally it has been inadvertently but harmlessly omitted from studies of heat convection in electrically conducting fluids under magnetic fields. In a fission or fusion rector (16), (18) and (19) would gain a heat-release term from the bombardment by neutrons or other particles or photons.

Boundary conditions

We have already remarked that, with homogeneous media, the thermoelectric e.m.f.'s show themselves only via boundary conditions at interfaces between media of different absolute thermoelectric power. The electric boundary condition must be generalized to allow for this.

Consider a fluid moving over a stationary conducting wall in the presence of a magnetic field with a component B_n, normal to the interface, directed into the fluid.

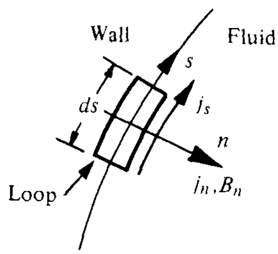


FIGURE 2. A portion of a wall/fluid interface.

Equation (1) may be integrated round the narrow loop of length ds shown in figure 2, straddling the interface. Viscosity requires the velocity to be strictly zero at the interface, but, where the Hartmann number based on B_n is large, it will often be appropriate to take the Hartmann layer so thin that it lies wholly within the loop, i.e. to allow non-zero velocities to persist virtually up to the wall. Let v_t be the tangential velocity component normal to the plane of figure 2, out of the paper.

In the integral of (1), $\oint \mathbf{E}.d\mathbf{r}$ contributes negligibly, for the area integral of (13) is a second-order small quantity, and so

$$\frac{j_{sw}}{\sigma_w} - \frac{j_s}{\sigma} + \frac{\partial}{\partial s}(\tau j_n) + v_t B_n = P\frac{\partial T}{\partial s} = \frac{\partial \mathscr{E}}{\partial s}, \tag{20}$$

in which n and s denote normal and tangential components in the plane of figure 2, w denotes wall quantities and τ is the contact resistance (if any) of unit area of interface. In order to make the interface temperature T unambiguous we exclude *thermal* contact resistance (the thermoelectric consequences of which appear to be largely unexplored). Again P is the thermoelectric power of the pair of metals, $S - S_w$, and $\mathscr{E}$ is their Seebeck e.m.f. $\int P dT$, relative to some datum temperature. A further condition similar to (20) could also be written down in the plane perpendicular to figure 2 and containing the n direction.

Another boundary condition which is altered by the advent of thermoelectricity is the thermal one which constrains temperature gradients normal to the interface. Again we exclude thermal contact resistance. Since the interface can have no heat capacity, the energy inputs and outputs per unit area are in balance and hence, from (6), we have

$$\tau j_n^2 + S_w T j_n - K_w \frac{\partial T_w}{\partial n} = S T j_n - K\frac{\partial T}{\partial n},$$

or

$$K_w \frac{\partial T_w}{\partial n} - K\frac{\partial T}{\partial n} = (S_w - S)\, T j_n + \tau j_n^2. \tag{21}$$

The term $(S_w - S)\, T j_n$ describes the *Peltier effect*, which arises because the ability of the current to transport heat changes abruptly at the interface. The normal gradients of temperature have to adjust when j_n is introduced, even if the dissipation associated with contact resistance τ is absent. In arriving at (21) we have tacitly used the boundary condition that j_n is continuous, in line with (8). The quality $T(S_w - S)$ is called the Peltier coefficient.

All other boundary conditions remain exactly as they are in ordinary MHD and heat-transfer problems.

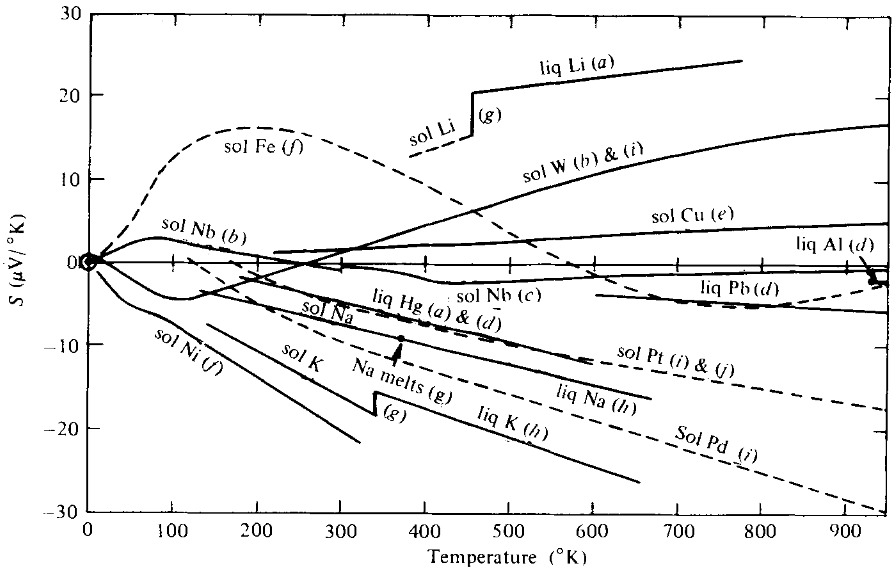


FIGURE 3. Some absolute thermoelectric powers for pure metals. (N.B. Small amounts of additives can produce large changes.) sol = solid; liq = liquid; Al = aluminium; Cu = copper; Fe = iron; Hg = mercury; K = potassium; Li = lithium; Na = sodium; Nb = niobium; Ni = nickel; Pb = palladium; Pt = platinum; W = tungsten. Sources: (*a*) Ioannides *et al.* (1975); (*b*) Carter *et al.* (1970); (*c*) Raag & Kowger (1965); (*d*) Marwaha (1967); (*e*) Marwaha & Cusack (1966); (*f*) Blatt *et al.* (1967); (*g*) Cusack & Enderby (1960); (*h*) Bradley (1962); (*i*) Cusack & Enderby (1958); (*j*) Kaye & Laby (1972, p. 48).

3. Practical magnitudes

Figure 3 presents some data on the absolute thermoelectric power S of various metals. Lithium is seen to be unusual in having a large and positive power whereas most metals have negative powers. The Seebeck effect, our main concern here, is therefore strong when liquid lithium adjoins a solid metal with negative power, such as stainless steel or niobium. It is noteworthy that lithium is so different thermoelectrically from its fellow alkali metals, sodium and potassium. It is like most metals, however, in having a significant change in power on melting (in contrast to sodium, which has none).

To establish the general order of magnitude of TEMHD effects, consider the flow of lithium along a metal duct of rectangular cross-section under a transverse magnetic field B parallel to one pair of walls and a uniform transverse temperature gradient $\partial T/\partial s$ parallel to the other pair of walls. Let there be no pressure gradient. If we neglect viscous (or turbulent) drag on the basis that the Hartmann number is very high, the fluid accelerates until $\mathbf{v}\times\mathbf{B}$ e.m.f.'s balance the Seebeck e.m.f. and all current flow ceases, for no streamwise $\mathbf{j}\times\mathbf{B}$ force is required in the steady state. (A fuller discussion of such flows appears later.) Then the boundary condition (20) indicates that the velocity is given by the equation

$$v = \frac{P}{B}\frac{\partial T}{\partial s}. \tag{22}$$

Note that the conductivity of the liquid or walls and the contact resistance have become immaterial, although their values would determine the time taken to reach the steady state. An upper constraint on the impedance of the walls emerges in the next section.

It should be noted that (22) contains B in the denominator, i.e. *higher* fields imply *lower* velocities, as Osterle & Angrist (1964) remarked. One way of looking at this is to regard the current-free condition as the superposition of pressure-driven flow in a conducting pipe, where the drag varies like vB^2, and TEMHD pumping, where the propulsive force varies like B, a lower power of B. This view of the matter also throws light on why the impedance of the current circuit (via liquid, walls and contact resistance) is immaterial, for it enters in the same way into both the drag and the propulsion. The fact that (22) indicates that $v \to \infty$ as $B \to 0$ is spurious, for the Hartmann number would have long ceased to be high in this limit.

Inserting representative magnitudes for

$$P = 25\,\mu\mathrm{V/K}, \quad B = 1\,\mathrm{tesla} \quad \text{and} \quad \partial T/\partial s = 10^4\,^\circ\mathrm{K/m}$$

(i.e. the rather high gradient of 100 °C/cm, chosen to show how strong the effect could be) gives velocities of 25 cm/s, a value which is definitely in the range of practical interest. Velocities of this order have been mentioned in studies of the problem of circulating lithium coolant in fusion reactors (e.g. Hunt & Hancox 1971). The heat flow rate with this temperature gradient is 0·5 MW/m^2, which is of the same order as the bremstrahlung, alpha-particle and other energy being deposited in the first wall of a highly rated fusion reactor. Notice however that in this case the heat is flowing *across* the lithium, which is moving parallel to the first wall and is not itself acting as a coolant. Devising geometries in which the lithium is both propelled by TEMHD and acts as a coolant itself is easier said than done!

In contrast to the Seebeck effect, the Peltier and Thomson effects turn out to be negligible in the context of liquid metals, so the corresponding extra terms can be left out of the differential equation (19) and the boundary condition (21) which govern the heat-transfer aspect of the problem. As is well known, the same is true of the viscous and Ohmic dissipation terms in most problems. The reaction of the MHD fields back on to the heat and temperature distributions is therefore confined purely to the effect whereby the heat convection is altered by changes in the fluid mechanics due to $\mathbf{j} \times \mathbf{B}$ forces. The mutual coupling between the thermal and MHD fields is thereby greatly simplified.

As evidence for these assertions, consider again fluid in a rectangular duct under conditions where the current flow is maximized by assuming negligible impedance in the walls (so that only the resistance of the liquid need be included) and by having the fluid at rest. Then (20) indicates current densities in the fluid equal to $j = -\sigma P\,\partial T/\partial s$, for figure 2 could represent part of the side wall of a rectangular duct whose top and bottom walls were maintained at different temperatures. At such a wall the Peltier heat rate per unit area would be of order $-PTj$ or $\sigma P^2 T\,\partial T/\partial s$. This has to be compared with the prevailing thermal conduction rate, of order $K\,\partial T/\partial s$. Their ratio $\sigma P^2 T/K$ takes the small value of 0·02, if we insert the magnitudes

$$P \approx 25\,\mu\mathrm{V/K}, \quad \sigma \approx 2\times 10^6\,\mathrm{mho/m}, \quad K \approx 50\,W/\mathrm{m}\,^\circ\mathrm{K} \quad \text{and} \quad T \approx 800\,^\circ\mathrm{K},$$

appropriate to lithium in a fusion reactor.

If the temperature difference between the top and bottom walls is ΔT, the total Thomson heat rate over the whole depth per unit area of top wall may be estimated from (10) as $jT(dS/dT)\,\Delta T$. The ratio of this to the Peltier heat is of order

$$\left(\frac{dS}{dT}\right)\frac{\Delta T}{P},$$

which numerically equals 0·1 if we insert the typical values

$$\Delta T \approx 200\,°\mathrm{K} \quad \text{and} \quad dS/dT \approx 13\times10^{-9}\,\mathrm{V}/°\mathrm{K}^2$$

(see figure 3). We shall henceforth neglect the Peltier effect and the even smaller Thomson effect. By the same token we can ignore the effect on the heat flow of the small fractions that are converted into hydraulic power in TEMHD pumping applications, where the pressure rises downstream. The thermal efficiency of such a heat engine is virtually zero, and the caloric theory of heat can stage a late rally!

4. Some simple standard problems in TEMHD duct flow

Hartmann flow

We have remarked that in thermoelectrically driven duct flow without a pressure gradient the velocity is *inversely* proportional to the transverse field B when viscous drag is negligible at high Hartmann number. At the other extreme, when B is so low that the $\mathbf{v}\times\mathbf{B}$ e.m.f. is unimportant compared with the Seebeck e.m.f., which then determines the currents, the essential balance is between $\mathbf{j}\times\mathbf{B}$ and viscous forces. We should therefore expect the velocity to be *directly* proportional to B at low Hartmann number in the absence of a pressure gradient. This is confirmed by the extension of the classic Hartmann problem which follows.

Consider rectilinear z-wise laminar motion between conducting parallel planes $x = \pm a$ under the influence of an x-wise, uniform, transverse field B and a uniform temperature gradient dT/dy. Let each wall have a thickness t and the fluid have viscosity η. The motion is assumed to be fully developed, with all variables a function of x only, apart from T and U [see (4)], which vary linearly with y, and the pressure p, which may vary linearly with z. We take P and K constant for simplicity. Heat flows purely in the y direction and there is no heat convection, for $DT/Dt = 0$. The fluid velocity v vanishes at the walls. The Hartmann number M is $Ba(\sigma/\eta)^{\frac{1}{2}}$.

If j denotes the y-wise current in the fluid, and j_s its value at either wall, Ohm's law requires that

$$j = \sigma(Bv - dU/dy) = \sigma Bv + j_s. \tag{23}$$

The streamwise force balance is expressed by

$$jB = \eta\, d^2v/dx^2 - dp/dz.$$

Hence
$$\eta\, d^2v/dx^2 - \sigma B^2 v = Bj_s + dp/dz = \text{constant},$$

from which
$$v = -\frac{1}{\sigma B^2}\left(Bj_s + \frac{dp}{dz}\right)\left[1 - \frac{\cosh(Mx/a)}{\cosh M}\right], \tag{24}$$

i.e. the usual Hartmann velocity profile. The mean velocity is given by

$$v_m = -\frac{1}{\sigma B^2}\left(Bj_s + \frac{dp}{dz}\right)\left(1 - \frac{\tanh M}{M}\right). \tag{25}$$

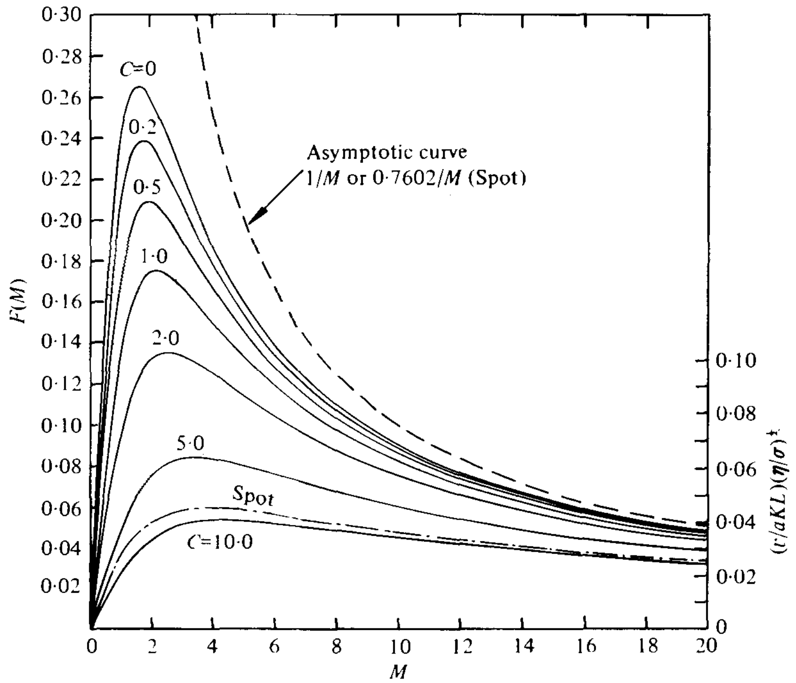


FIGURE 4. The function $F(M)$ for Hartmann flow; *also* the maximum velocity due to an impurity spot. (The curves designated 'Spot' and the right-hand ordinate scale apply.) $C = at\sigma/t\sigma_w$.

The mean current density in the fluid [from (23)] is

$$j_m = \sigma B v_m + j_s = -(t/a) j_w, \tag{26}$$

for all the current returns via the walls at an intensity j_w, say, if we assume that there is no other external return circuit. Moreover, (20), with v and j_n set to zero, gives

$$\frac{j_w}{\sigma_w} - \frac{j_s}{\sigma} = P\frac{dT}{dy}. \tag{27}$$

Eliminating j_s and j_m from (25)–(27) gives

$$v_m = \frac{M - \tanh M}{M + C\tanh M}\left(\frac{P}{B}\frac{dT}{dy} - \frac{1+C}{\sigma B^2}\frac{dp}{dz}\right), \tag{28}$$

in which $C = a\sigma/t\sigma_w$, a measure of wall impedance in comparison with that of the fluid. In the case where all flow is prevented (i.e. a TEMHD pump at standstill) the rising pressure gradient is

$$\frac{dp}{dz} = \frac{\sigma BP}{1+C}\frac{dT}{dy}, \tag{29}$$

independently of M (for η is irrelevant if the fluid is at rest).

If we turn next to the case of no pressure gradient, (28) becomes

$$v_m = Pa\left(\frac{\sigma}{\eta}\right)^{\frac{1}{2}}\frac{dT}{dy}F(M), \quad \text{where} \quad F(M) = \frac{M - \tanh M}{M(M + C\tanh M)}. \tag{30}$$

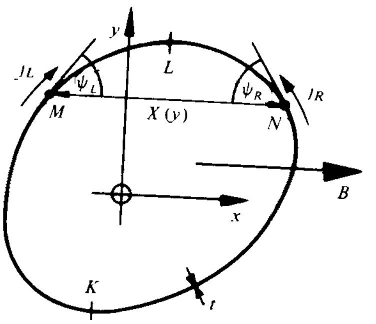


FIGURE 5. Cross-section of a conducting duct of arbitrary shape. The velocity v is out of the paper.

The function F is plotted against M for various values of C in figure 4. The transition from direct to inverse proportionality to M or B is clearly revealed. The case $C = 0$ corresponds to zero wall resistance and generates the greatest flow rate. For M large, F essentially equals $1/(M+C)$, in which M and C respectively represent the relative impedances of the Hartmann layers and walls. In a fusion-reactor blanket, C would probably be fairly large (*c*. 10) because duct walls would be kept as thin as possible to avoid loss of neutrons capable of breeding, but M would be much larger (*c*. 10^3–10^4). It is fortunate that endeavours to get the lithium to pump itself across the magnetic field with negligible pressure gradients do not depend on providing walls of low impedance.

In subsequent work we shall assume that the magnetic field is large enough for the condition $M \gg C$ to be satisfied, and then $F \approx 1/M$ and $v_m \propto 1/B$ (the characteristic high-M regime). Under these conditions, the Hartmann layers (and viscous effects generally) can be ignored in the application of boundary condition (20). When M is large compared with both C and unity, a negligibly small extra thermoelectric current is driven between the walls and Hartmann layers, sufficient to overcome the relatively weak viscous drag.

Two-dimensional duct flow problems (conducting walls)

More generally, one can consider fully developed laminar duct flows at high M where the transverse current distribution is two-dimensional. Figure 5 illustrates the general case. We confine attention to cases where the imposed, transverse magnetic field B is uniform and in the x direction, the fluid is in electrical contact with all parts of the wall, without contact resistance, and the wall thickness t is assumed constant and small compared with a typical duct dimension a. This avoids having to find matching solutions of Laplace's equation in the conducting walls. Since the flows are fully developed with no streamwise variation, there is no heat convection, and the thermal side of the problem is just as if the fluid were stationary. We shall assume that this part of the problem has been solved and the prevailing interface temperature and corresponding value of $\mathscr{E}$ are known as functions of position along the periphery of the fluid.

We continue to assume that M is large enough for viscosity to be neglected. At points K and L, the boundary layers will be of thickness of order $1/M^{\frac{2}{3}}$ (cf. Roberts 1967) rather than $1/M$. If their impedance is still high compared with that of the walls we

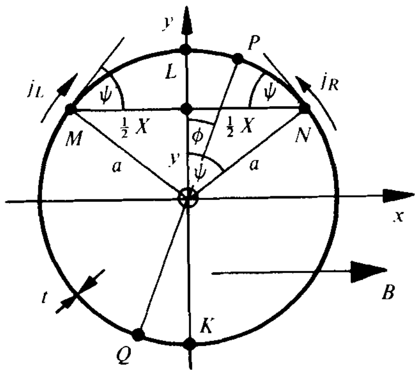


FIGURE 6. Cross-section of a circular duct.

can ignore them. If the periphery has straight portions of finite length parallel to the field, the boundary layers there will have a thickness of order $1/M^{\frac{1}{2}}$ (cf. Hunt & Shercliff 1971). We do not consider such cases in this paper.

In the absence of acceleration or viscous stress, (12) indicates that

$$-j_y B = dp/dz = \text{constant},$$

and (8) then implies that $\partial j_x/\partial x = 0$, i.e. $j_x = j_x(y)$. The curl of Ohm's law in the form $\mathbf{j}/\sigma = \mathbf{v} \times \mathbf{B} - \operatorname{grad} U$ indicates that

$$\sigma B\, \partial v/\partial x = -dj_x/dy, \tag{31}$$

i.e. v varies linearly with x at each value of y and

$$\sigma B(v_R - v_L)/X(y) = -dj_x/dy, \tag{32}$$

in which the subscripts R and L indicate values at the right and left boundaries in figure 6 and $X(y)$ is $x_R - x_L$. These thermoelectric problems are seen to include cases where vorticity components normal to the imposed field are not suppressed, cf. those mentioned by Shercliff (1975) elsewhere.

We may neglect variation in j_L and j_R, the tangential current densities within the respective walls at each value of y, for they are large compared with the normal current densities if $t \gg a$. Then current conservation requires that

$$t(j_L + j_R) + j_y X = 0, \tag{33}$$

there being no other external circuit,

and
$$t\, dj_R/dy = j_x + j_y \cot \psi_R. \tag{34}$$

The boundary condition (20), applied respectively at M and N, gives

$$\frac{j_L}{\sigma_w} - \frac{1}{\sigma}(j_x \cos\psi_L + j_y \sin\phi_L) + v_L B \sin\psi_L = \frac{d\mathscr{E}_L}{dy}\sin\psi_L \tag{35a}$$

and
$$\frac{j_R}{\sigma_w} - \frac{1}{\sigma}(-j_x \cos\psi_R + j_y \sin\psi_R) + v_R B \sin\psi_R = \frac{d\mathscr{E}_R}{dy}\sin\psi_R, \tag{35b}$$

which combine with (32) to yield

$$\frac{1}{\sigma_w}\left(\frac{j_R}{\sin\psi_R} - \frac{j_L}{\sin\psi_L}\right) - \frac{d}{dy}\left(\frac{j_x X}{\sigma}\right) = \frac{d}{dy}(\mathscr{E}_R - \mathscr{E}_L). \tag{36}$$

This relation may be written directly as the line integral of Ohm's law (1) taken round a loop consisting of MN and a parallel line dy higher, joined by two short arcs of wall. Then j_R and j_L may be eliminated from (33), (34) and (36) so as to provide a second-order, linear, differential equation in j_x which may be solved subject to the condition that j_x must be finite at K and L. All other quantities are then readily deduced.

Circular ducts

As a general treatment is fairly complicated, in order to bring out the physics clearly we take the markedly simpler case of a duct of circular cross-section of radius a. (See figure 6.) Now $\psi_L = \psi_R = \psi$, say, and $y = a\cos\psi$ while $X = 2a\sin\psi$. It is now convenient to Fourier analyse $\mathscr{E}$, expressed as a function of ψ, measured clockwise from Oy, into even and odd harmonics:

$$\mathscr{E} = \mathscr{E}_m + \Sigma A_n \cos n\psi + \Sigma B_n \sin n\psi, \quad n = 1, 2, \ldots, \infty. \tag{37}$$

Now (36) becomes

$$\frac{a}{2\sigma_w}(j_R - j_L) + a\frac{d}{d\psi}\left(j_x \frac{\sin\psi}{\sigma}\right) + \Sigma n B_n \cos n\psi = 0, \tag{38}$$

which together with (33) and (34) yields an equation for j_x:

$$\frac{d^2}{d\psi^2}(j_x \sin\psi) - C j_x \sin\psi = \frac{\sigma}{a}\Sigma n^2 B_n \sin n\psi, \quad \left(C = \frac{\sigma a}{t\sigma_w}\right), \tag{39}$$

from which

$$j_x \sin\psi = -\frac{\sigma}{a}\Sigma \frac{B_n \sin n\psi}{C/n^2 + 1}. \tag{40}$$

The complementary function must vanish, for j_x is finite at $\psi = 0$ or π. It is not surprising to find that the transverse currents j_x are determined by the odd part of $\mathscr{E}$, which measures the asymmetry of the $\mathscr{E}$ distribution. Equation (40) is essentially a statement that $j_x(y)$ is a polynomial in y. From this, $\partial v/\partial x$ can then be evaluated from (31). It is inversely proportional to B. The velocity itself is completely determined if we identify v_0, the value of v on the y axis, by adding (35a, b) and using (33) so as to get

$$Bv_0 = \frac{1}{2}\frac{d}{dy}(\mathscr{E}_R + \mathscr{E}_L) + \frac{j_y}{\sigma}(1 + C) \tag{41}$$

$$= \frac{\Sigma n A_n \sin n\psi}{a\sin\psi} + \frac{j_y}{\sigma}(1 + C), \tag{42}$$

which reveals that v_0 also is a polynominal in y. The velocity profile can be regarded as essentially a superposition of the last, constant term, which corresponds to Chang & Lundgren's (1961) pressure-driven solution for isothermal flow in a conducting circular pipe with $M \gg C$ (for which the velocity is uniform), upon TEMHD flow without a pressure gradient (for which $j_y = 0$). Let us consider further this latter case, in which all velocities are inversly proportional to B as usual.

If $\mathscr{E}$ is symmetrically distributed, j_x also vanishes and we have again the pure situation in which $\mathbf{v} \times \mathbf{B}$ and Seebeck e.m.f.'s can balance, The boundary condition (20), with no currents, determines the same value of v at both M and N and, in the usual manner characteristic of many high-M duct flows, v also takes this value at all points along the field line through M and N. However, if $\mathscr{E}$ is asymmetrical, (20) without currents would produce conflicting values for v at M and N. In consequence the more complicated process described by (38)–(40) ensues. The first term in (41) indicates how v_0 strikes a compromise between the two boundary conditions at M and N.

Though the antisymmetrical part of $\mathscr{E}$ controls the velocity via $\partial v/\partial x$, it does not affect the value of v_0 and obviously produces no contribution to the overall flow rate. In fact the mean velocity v_m over the pipe cross-section is given by

$$v_m = \int \frac{v_0 X\, dy}{\pi a^2}, \tag{43}$$

for v varies linearly with x, or by

$$\begin{aligned} v_m &= \left(-\frac{dp}{dz}\right)\left(\frac{1+C}{\sigma B^2}\right) + \frac{2}{\pi Ba}\int_0^\pi \Sigma n A_n \sin n\psi \sin \psi \, d\psi \\ &= \left(-\frac{dp}{dz}\right)\left(\frac{1+C}{\sigma B^2}\right) + \frac{A_1}{Ba}, \end{aligned} \tag{44}$$

i.e. only the first even harmonic of (37) contributes to the overall flow. By the same token, if the flow is blocked, the resulting pressure gradient is determined purely by this harmonic.

It should also be noted that, in the above treatment, we have tacitly assumed that the high harmonics of $\mathscr{E}$ are sufficiently weak for the associated velocity gradients not to rise to a level where the inviscid assumption fails.

Consider the special case where only first harmonics are present and

$$\mathscr{E} = \mathscr{E}_m + D\cos(\psi - \phi), \text{ say,}$$

where ϕ is defined in figure 7, P and Q being the points of maximum and minimum temperature. We then find that

$$j_x = -\frac{D\sigma \sin\phi}{a(1+C)}, \tag{45}$$

i.e. j_x takes a *uniform* value, as does j_y. The uniform current in the fluid flows obliquely. In this exceptional case, $\partial v/\partial x$ vanishes and v takes the uniform value

$$v = \frac{D\cos\phi}{Ba} + \left(\frac{dp}{dz}\right)\left(\frac{1+C}{\sigma B^2}\right). \tag{46}$$

The TEMHD contribution falls from a maximum to zero as ϕ rises from 0 to $\frac{1}{2}\pi$, as would be expected in view of the tendency of the thermoelectric currents to flow parallel to PQ in stationary fluid. At a standstill, dp/dz is proportional to $\cos\phi$ in fact, i.e. to the component of $\mathbf{j}$ normal to $\mathbf{B}$. In the absence of dp/dz, however, the currents flow parallel to $\mathbf{B}$ under the influence of the $\mathbf{v}\times\mathbf{B}$ e.m.f.'s.

This simple behaviour only occurs when a circular pipe is combined with a purely sinusoidal $\mathscr{E}$ distribution and a uniform transverse field, however.

Some interesting stability questions must arise from the complex velocity profiles that occur, particularly when $\mathscr{E}$ has asymmetry. If B were lowered its stabilizing powers would fall and meanwhile the thermoelectrically driven velocities would rise like $1/B$. It is probable that steady, thermoelectrically driven turbulence could occur, provided the peripheral temperature distribution could be maintained in the face of the changed heat transfer (somewhat enhanced despite the low Prandtl number). On the other hand, the nature of the heat sources and sinks might be such that turbulent heat transfer, in lowering the peripheral temperature differences, caused subsequent de-excitation of the turbulence and an intermittent or cyclic state might occur. It would be important thoroughly to understand such phenomena as a safeguard against possible thermal fatigue in fusion-reactor blankets, wherever liquid lithium was contained in pipes or closed containers.

Ducts with 'mixed' walls

All the phenomena discussed by Hunt & Shercliff (1971) that occur when walls are wholly or partially non-conducting can be generalized to include TEMHD effects. Here we explore merely the case where, in figure 5, the right-hand boundary is a non-conducting wall or a free surface (which would be plane).

As a result, (33) becomes $tj_L + j_y X = 0$, which determines j_L, (38) becomes

$$j_x + j_y \cot \psi_R = 0,$$

which determines j_x, and (35a) then determines v_L directly. In fact

$$Bv_L = \frac{d\mathscr{E}_L}{dy} + \frac{j_y X}{t\sigma_w \sin \psi_L} + \frac{j_y}{\sigma}(1 - \cot \psi_R \cos \psi_L). \tag{47}$$

The last term vanishes when the walls at M and N are perpendicular. The second term on the right is constant for a circular duct. Meanwhile the variation of v with x is determined by (31). There is no such variation if dp/dz and j_y vanish or if ψ_R is constant, i.e. the right-hand wall or free surface is plane. Under these conditions the velocity at all points along each particular field line is constant and is determined purely by the Seebeck boundary condition (20) at the point where the line cuts the one, thermo-electrically active interface. More generally, (31) and (47) indicate that at each value of y, the velocities are determined purely by the various conditions that prevail along the particular field line, including the curvature of the non-conducting wall.

5. TEMHD in industrial metallurgy

It is reasonable to look for applications of TEMHD in metallurgical processes in view of the high temperature gradients that are commonly present. In existing applications of MHD to metallurgy, other than those where a high current is imposed directly (as in arc furnaces, welding, aluminium smelting, etc.), it is normally necessary to induce the current by the use of alternating fields and then there are difficulties because skin effect inhibits their penetration, particularly when the liquid metal is shielded by solidified metal as in casting processes. If the current could be of *thermoelectric* origin, then steady magnetic fields could be used and the problem of poor penetration could be circumvented. The most appropriate goal of applying MHD in this way would appear to be to promote stirring so as to avoid or disperse non-uniformities or to break up incipient dendrites.

At first sight the advancing interface between the solid and liquid phase as a melt solidifies would appear to be a ready source of thermoelectric effects, because most metals change their thermoelectric power on melting, but this is foiled by the fact that the phase boundary is normally isothermal, i.e. the peripheral temperature gradient, vital in condition (20), is lacking. A more promising source of thermoelectric activity occurs when there is a variation of thermoelectric power owing to segregation of impurities. If there is also a temperature gradient and an imposed magnetic field of suitable magnitude and orientation, it is conceivable that desirable TEMHD stirring of the molten metal could occur in the vicinity. There could be automatic dispersion of patches of impurity as they formed and an improved quality of the product. This is highly speculative, as little information is available on the likely magnitudes of thermoelectric effects of this kind, but it appears to be worth pursuing further in view of the benefits it might offer. Some experimental data are badly needed.

Here we confine ourselves to solving a fairly simple problem as a preliminary indication of some of the theoretical possibilities. It is a case with high symmetry, which allows an analytical solution, and in fact results in no dispersal of the postulated non-uniformity of composition, but it is probable that stirring would occur in less symmetrical configurations. It typifies those cases in which the thermoelectric action is due to *continuous* variation of material composition rather than a sharp interface.

We assume that there is a spherically symmetric distribution of S, centred on the origin, and that there are a uniform magnetic field B and a uniform temperature gradient, parallel to each other, in the z direction, say. The implicit assumption is that the variation of material composition does not affect the thermal conductivity. We assume also that the viscosity and electrical conductivity are uniform. The problem, being axisymmetric, is conveniently treated in cylindrical polar co-ordinates (r, θ, z).

The active term, $\operatorname{grad} S \times \operatorname{grad} T$, in the curl of Ohm's law is an azimuthal vector, as is $\operatorname{curl} \mathbf{j}$, so the currents circulate in meridional planes, producing azimuthal $\mathbf{j} \times \mathbf{B}$ forces and azimuthal velocities which do not produce any thermal convection effects. The resulting $\mathbf{v} \times \mathbf{B}$ e.m.f.'s are in meridional planes and modify the current pattern. We neglect inertial effects, thereby suppressing centrifugally driven secondary motions in meridional planes, which would of course produce desirable stirring. It is convenient to use the azimuthal induced field $\mathbf{H}$ as a vector potential for the currents, with $\mathbf{j} = \operatorname{curl} \mathbf{H}$, but we ignore its effect on the imposed field.

The curl of Ohm's law (1) then becomes

$$\operatorname{curl}\operatorname{curl} \mathbf{H}/\sigma = B(\partial/\partial z)\,\mathbf{v} - \operatorname{grad} S \times \operatorname{grad} T \tag{48}$$

and the balance between viscous and $j \times \mathbf{B}$ forces is expressed by

$$\eta \operatorname{curl}\operatorname{curl} \mathbf{v} = B(\partial/\partial z)\,\mathbf{H}, \tag{49}$$

there being no azimuthal pressure gradient. If we set

$$\left.\begin{aligned} \mathbf{w}_1 &= \mathbf{H}(\sigma\eta)^{-\frac{1}{2}} + \mathbf{v} \\ \text{and} \qquad \mathbf{w}_2 &= \mathbf{H}(\sigma\eta)^{-\frac{1}{2}} - \mathbf{v}, \end{aligned}\right\} \tag{50}$$

then $\mathbf{w}_1$ obeys the decoupled equation

$$-\operatorname{curl}\operatorname{curl} \mathbf{w}_1 + \left(\frac{\sigma}{\eta}\right)^{\frac{1}{2}} B \frac{\partial \mathbf{w}_1}{\partial z} = \left(\frac{\sigma}{\eta}\right)^{\frac{1}{2}} \operatorname{grad} S \times \operatorname{grad} T, \tag{51}$$

while $\mathbf{w}_2$ can be found by changing the sign of z. To represent a diffuse spot of impurity we shall take the particular distribution

$$S = K \exp\left[-(r^2 + z^2)/a^2\right] + S_\infty, \tag{52}$$

which defines the length scale a, upon which the Hartmann number $M = Ba(\sigma/\eta)^{\frac{1}{2}}$ can be based. Let $|\text{grad}\, T| = L$. Then, if $w_1 = |\mathbf{w}_1|$, (51) yields

$$\frac{\partial^2 w_1}{\partial z^2}+\frac{\partial^2 w_1}{\partial r^2}+\frac{1}{r}\frac{\partial w_1}{\partial r}-\frac{w_1}{r^2}+\left(\frac{\sigma}{\eta}\right)^{\frac{1}{2}} B\frac{\partial w_1}{\partial z} = \frac{2KL}{a^2}\left(\frac{\sigma}{\eta}\right)^{\frac{1}{2}} r\exp\left[-(r^2+z^2)/a^2\right]. \tag{53}$$

The boundary conditions are that $\mathbf{v}$, $\mathbf{H}$ and so $\mathbf{w}_1$ and $\mathbf{w}_2$ vanish on the axis ($r = 0$) and tend to zero far from the origin. It proves possible to satisfy all conditions by taking

$$w_1 = 2KL(\sigma/\eta)^{\frac{1}{2}} rU_1(R)\exp\left(\tfrac{1}{16}M^2 - Mz/2a\right), \tag{54}$$

in which the dimensionless quantity U_1 is a function purely of R, where

$$R^2 = \left(\frac{z}{a}-\frac{M}{4}\right)^2+\frac{r^2}{a^2}, \tag{55}$$

i.e. R is a dimensionless radius vector measured from an offset origin! U_1 must satisfy the ordinary differential equation

$$\frac{d^2U_1}{dR^2}+\frac{4}{R}\frac{dU_1}{dR}-\left(\frac{M}{2}\right)^2 U_1 = \exp(-R^2), \tag{56}$$

together with the conditions that $RU_1 \to 0$ as $R \to 0$, and that, as $R\to\infty$, $RU_1 \to 0$ faster than $e^{-\frac{1}{2}MR}$ in order that $w_1 \to 0$ at large negative z despite the exponential factor in (54). The solution is

$$U = \frac{1}{4R^2}\left\{\exp(-R^2) - \frac{\pi^{\frac{1}{2}}\exp\left(\frac{1}{16}M^2\right)}{4R}\left[(1+\tfrac{1}{2}MR)\,e^{-\frac{1}{2}MR}\operatorname{erfc}\left(\tfrac{1}{2}M-R\right)-(1-\tfrac{1}{2}MR)\right.\right.$$
$$\left.\left.\times\, e^{\frac{1}{2}MR}\operatorname{erfc}\left(\tfrac{1}{4}M+R\right)\right]\right\}, \tag{57}$$

from which w_1 follows. As $R \to 0$, $U_1 \to (4M^2-32-\frac{1}{192}\pi^{\frac{1}{2}}M^3\exp\left(\frac{1}{16}M^2\right)\operatorname{erfc}\left(\frac{1}{2}M\right)$. w_2 can be generated by a change of sign of z, and then $|\mathbf{v}|$ and $|\mathbf{H}|$ can be found from

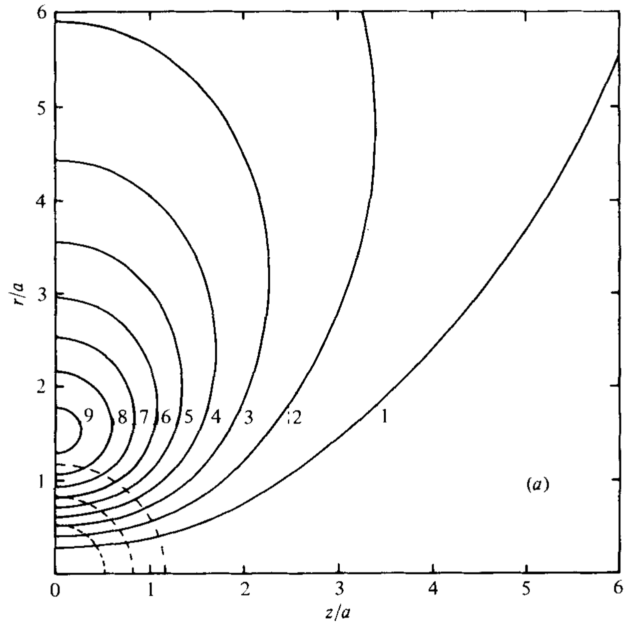
$$|\mathbf{v}| = \tfrac{1}{2}(w_1-w_2), \quad |\mathbf{H}| = \tfrac{1}{2}(w_1+w_2)(\sigma\eta)^{\frac{1}{2}}. \tag{58}$$

Figures 7 and 8 show some typical results. The broken lines are the circles on which $S-S_\infty$ equals $\frac{1}{4}K$, $\frac{1}{2}K$ or $\frac{3}{4}K$. The current field lines are contours upon which $r|\mathbf{H}|$ is constant. As M rises, the disturbance becomes increasingly extended in the direction of the magnetic field as one would expect. As B varies, the maximum velocity $0{\cdot}044997\; aKL(\sigma/\eta)^{\frac{1}{2}}$ occurs at $z/a = 1{\cdot}193$, $r/a = 0{\cdot}996$ when $M = 3{\cdot}863$. Figure 4 also shows how the maximum velocity varies as B varies and compares it with the case of Hartmann flow. Once again there is a transition from variation like B (at low M) to variation like B^{-1} (at high M).

When $M \gg 1$, the absence of significant viscous drag removes the need for currents and so $\mathbf{v}\times\mathbf{B}$ balances the Seebeck e.m.f.'s, i.e. the left-hand side of (48) can be set to zero. The resulting solution is simply

$$|\mathbf{v}| = \frac{KL\pi^{\frac{1}{2}}(r/a)\exp\left[-(r/a)^2\right]}{B}\operatorname{erf}(z/a), \tag{59}$$

which implies a disturbance that persists indefinitely at large $|z|$. In fact viscous effects would always attenuate the disturbance ultimately, however large M was. From (59) the maximum value of $|\mathbf{v}|/aKL(\sigma/\eta)^{\frac{1}{2}}$ is $0{\cdot}7602/M$ at large z and $r/a = 2^{-\frac{1}{2}}$, and this



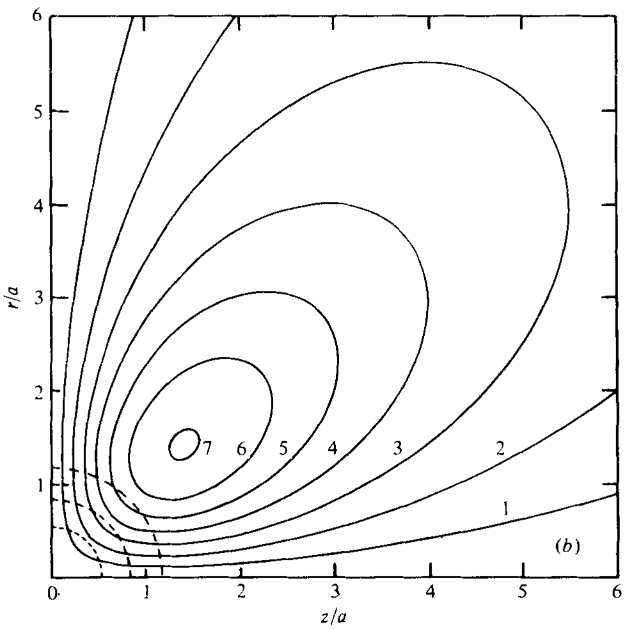


FIGURE 7. TEMHD due to an impurity spot at $M = 0$: (a) current pattern and (b) contours of constant velocity. The values shown on the contours are

$$(a)\ -40r|\mathbf{H}|/\sigma a^2KL \quad \text{and} \quad (b)\ 200\eta|\mathbf{v}|/\sigma a^2BKL.$$

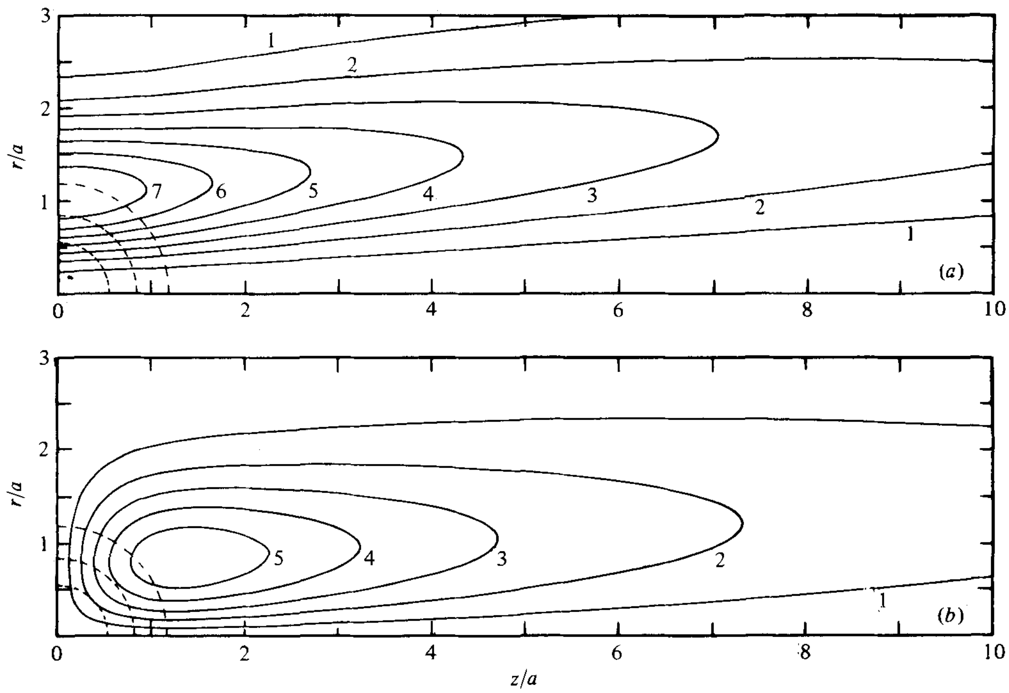


FIGURE 8. TEMHD due to an impurity spot at $M = 15$: (a) current pattern and (b) contours of constant velocity. The values shown on the contours are

$$(a)\ -200r|\mathbf{H}|/\sigma a^2 KL \quad \text{and} \quad (b)\ 200|\mathbf{v}|/aKL(\sigma/\eta)^{\frac{1}{2}}.$$

asymptotic variation is also shown in figure 4. If dispersion of an impurity patch is the aim, there is no incentive to generate motion remote from it, however.

As $M \to 0$, the $\mathbf{v} \times \mathbf{B}$ e.m.f.'s become negligible and $\mathbf{H}$ is directly determinable from (48) (or from (57) with $M = 0$) as

$$|\mathbf{H}| = \frac{\sigma KLr}{4R^3}\{2R\exp(-R^2) - \pi^{\frac{1}{2}}\operatorname{erf} R\}. \tag{60}$$

Then from (49), or otherwise, we have

$$|\mathbf{v}| = \frac{B\sigma KLrz}{16\eta R^5}\{6R\exp(-R^2) + (2R^2-3)\pi^{\frac{1}{2}}\operatorname{erf} R\}. \tag{61}$$

Figure 7 shows this case. The velocity contours now have symmetry about the line $r = z$.

The oppositely directed swirls for $z \gtrless 0$ could lead to instability and hence to stirring, despite the axisymmetry of the unperturbed flow, although inertia forces would probably be small.

To establish plausible orders of magnitude, consider the following values (in S.I. units): $a = 10^{-3}$ (a small, 1 mm spot), $K = 10^{-6}$ (i.e. $1\,\mu V/^\circ$K), $L = 10^3$ (i.e. 10 °C/cm), $\sigma = 10^6$, $\eta = 10^{-3}$ (typical of liquid metals). The maximum swirl velocity v_1 is then $1{\cdot}5 \times 10^{-3}$ (1.5 mm/s), which is quite high, given the small scale.

At a convenient field level of $B = 0{\cdot}2$ (2000 gauss), $M = 6$ (i.e. not far from the maximum velocity point) and the interaction parameter $\sigma B^2 a/\rho v_1$ takes the large value of 10, approximately, if we choose $\rho \approx 3\times 10^3$ (e.g. aluminium). This implies that secondary velocities v_2 (in meridional planes) will be about one tenth of v_1, being opposed by $\mathbf{j}\times\mathbf{B}$ forces such that

$$\rho v_1^2/a \approx \sigma v_2 B^2.$$

Our neglect of inertial terms is thus seen to be a reasonable first approximation.

6. Concluding remarks

While we need be under no illusions that TEMHD is a subject of massive, central importance in science or technology, the fact that the order-of-magnitude calculations in §§ 3 and 5 and also preliminary experiments with mercury in copper indicate readily measurable and significant effects is sufficient justification for this new hybrid subject to receive some attention. This paper has indicated some of the possibilities. It should be noted again that we have confined ourselves here to cases where the temperature distribution is not affected by changed thermal convection due to TEHMD motion, or at least the crucial temperature distribution along a wall is known *ab initio*. The more demanding problems that involve a mutual coupling of the thermal and MHD fields still remain to be tackled. It is interesting to speculate whether there are geophysical applications of TEHMD also.

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